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Energy–momentum tensor in phase space: a connection between Schrödinger energy–momentum tensor and Terletsky distribution function

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Abstract. Schrödinger energy–momentum tensor for a matter field interacting with an external electromagnetic field is represented in phase space by means of Terletsky distribution function. The explicit form of this distribution function in the case of a hydrogen-type atom is found. The structure of the Schrödinger energy–momentum tensor in phase space is compared with the structure of a real fluid and its local pressure and local energy are presented in the fundamental state of a hydrogen-type atom. The negative values of the Terletsky distribution function are not rejected in view of their participation in the phase space energy–momentum tensor. The negative part of the quantum distribution function could be interpreted as phase space density for anti-constituents in a hypothetical sub-quantum level of the matter.

1. Introduction

Quantum distribution functions have been investigated since 1932 when Wigner [1] discovered his well-known distribution function for the purposes of quantum optics. Many other quantum distribution functions were found later, each one of them suggested to describe a particular branch of quantum physics. In 1966 Cohen [2] classified almost all the quantum distribution functions. The development of the basic ideas, the classifications and the main tendencies of investigations in the scope of the quantum distribution functions have been traced out by many authors [3–11].

It is a well-known fact that some of the quantum distribution functions have negative values in some domains in phase space. This fact has always complicated their interpretation as probability densities in phase space. Here we intend to show that the negative values are no great disadvantage as regards the possibility of including some of the quantum distribution functions in a phase space representations of the energy–momentum tensor of the system.

According to [12] and [13] we can decompose any quantum distribution function in the following way:

$$F(x,k)=F^{+}(x,k)-F^{-}(x,k) \tag{1}$$

where the functions

$$F^{\pm}(x,k)=(1/2)(|F(x,k)|\pm F(x,k)) \tag{1.2}$$

are always positive and could be considered as probability densities for pairs of random physical quantities (η^+, η^-). This suggestion for $F^{\pm}(x, k)$ could be related to Kruger's [14] idea to consider joint distribution functions in the sense of classical probability theory of a stochastic variable. It is important to note that the functions $F^{\pm}(x, k)$ in (1.2) are continuous and integrable if $F(x, k)$ is smooth and integrable.

In the above and in what follows $x=(x^\mu)$ denotes the position in Minkowski space and $k=(k_\mu)$ is the four-dimensional wavevector related to the four-momentum p_μ by $k_\mu = p_\mu/\hbar c$, $\mu = 0, 1, 2, 3$, where $\hbar$ is the reduced Planck's constant and c is the speed of light. For interpretation purposes in the sequel we accept Recami and Rodrigues [15] concept that special relativity is based on the whole proper group of both ortho- and anti-orthochronous Lorentz transformations, i.e. that in special relativity, particles as well as antiparticles are included.

Here we should mention that Vigier and Terletsky [16] have already discussed the possibility of representing the probability density of a system as a difference of two positive probability densities, corresponding to the particles and antiparticles forming the system. The representation (1.1) appears as a generalization of their concept to the case of phase space distribution functions.

The question about negative quasi-probabilities was also considered by Werner [9]. He evaluated the minimal value of the negative probability in the case of Wigner quantization of arrival time of a particle at the origin and the phase of a harmonic oscillator.

As far as we know the first successful attempt to relate the Wigner distribution function to current density, energy–momentum density and spin-density of a system was by Boer and van Weert [17]. De Groot *et al.* [18] showed that it was possible to represent the energy–momentum tensor of a system of scalar non-interacting particles as the second moment of the Wigner distribution function $F_W(x, k)$:

$$T_{\mu\nu}(x) = \text{constant} \int k_\mu k_\nu F_W(x, k)\, d^4 k.$$

Here and in what follows all integrations are carried out from $-\infty$ to $+\infty$, unless stated otherwise.

One observes that it is necessary to use an integration over a four-dimensional δ-function in order to introduce the Wigner distribution function in the energy–momentum tensor in equation (1.3) (see Chapter III in [18] for details).

Also, a phase space decomposition of the energy–momentum tensor of interacting scalar particles by means of Wigner distribution function has been applied by Cooper and Sharp [19] to account for pion production from a scalar source.

However, there exists another possibility that is more natural both from the mathematical and physical point of view—this is by means of Fourier expansion to obtain suitable representations of the current density, the energy–momentum tensor and the spin density in the phase space. In this case the quantum distribution function is the function first introduced by Terletsky [20] and later rediscovered by Margenau and Hill [21].

There are certain differences between the Wigner distribution function and the Terletsky distribution function. For example, the Wigner distribution function is obtained by means of a Fourier transform of the off-diagonal elements of the density operator [22] and maybe that is why it is suitable for description of the mixed states of many-particle systems. Since the Terletsky distribution function is found through

Fourier expansion of one of the two wavefunctions in quantum mechanical probability we suppose that it is more convenient for description of pure states of a single system.

So, the aims of the work are: (1) to find a phase space representation of the Schrödinger energy–momentum tensor [23] of a scalar matter field interacting with an external electromagnetic field and (2) to give an explicit expression of the Terletsky distribution function in the case of hydrogen-type atoms.

The organization of the work is as follows: in section 2 we transform the Schrödinger energy–momentum tensor in a way suitable to obtain its phase space representation. Then we find its relation to the Terletsky distribution function. In section 3 using Fourier analysis on Euclidean spaces [24] we obtain the Terletsky distribution function for hydrogen-type atoms in spherical coordinates, both in position and momentum spaces. In section 4 we make some comments concerning the results obtained in the previous sections. In Section 5 the local pressures, energies and longitudinal and transverse interactions in the fundamental state of hydrogen atoms are considered.

2. Relation between Schrödinger energy – momentum tensor for matter fields and the Terletsky distribution function

Consider the energy–momentum tensor for a scalar field Ψ interacting with an external electromagnetic field A_μ, $\mu=0, 1, 2, 3$, introduced by Schrödinger [23]

$$S_{\mu\nu}(x)=\alpha\,\mathrm{Re}[(D_\mu\Psi)^*(D_\nu\Psi)]-\mathscr{L}g_{\mu\nu} \tag{2.1}$$

where α is a normalizing constant, $D_\mu=\partial_\mu-\mathrm{i}eA_\mu$, $\mu=0, 1, 2, 3$, $\partial_\mu=\partial/\partial x^\mu$, e is the elementary electric charge, A_μ is the four-potential of the external electromagnetic field, $g_{\mu\nu}$ is the metric in Minkowski space and $\mathscr{L}$ is the Lagrangian of the system [23]:

$$\mathscr{L}=(\alpha/2)[(D_\sigma\Psi)^*(D^\sigma\Psi)-\mathrm{k}^2\Psi^*\Psi]. \tag{2.2}$$

Here Einstein's summation rule is understood and the wavenumber k is determined via the rest mass m_0 of the scalar field by the well-known relation

$$hc\mathrm{k}/(2\pi)=m_0c^2. \tag{2.3}$$

According to Schrödinger the field Ψ obeys the Klein–Gordon equation with external electromagnetic field

$$[D_\nu D^\nu+\mathrm{k}^2]\Psi=0. \tag{2.4}$$

In his thesis Anastassov [25] pointed out the possibility of transforming the Schrödinger energy–momentum tensor into phase space by the use of the Terletsky distribution function. Here a modified version of his proof and interpretation of the results is presented.

The first statement is that the energy–momentum tensor defined in (2.1) can be transformed into a form suitable for its transition into phase space, i.e.

$$S_{\mu\nu}(x)=-\alpha\,\mathrm{Re}[\Psi^*(D_\mu D_\nu\Psi)]+(\alpha/2)[\partial_\mu\partial_\nu-(1/2)g_{\mu\nu}\Box](\Psi^*\Psi). \tag{2.5}$$

where $\Box=\partial_\sigma\partial^\sigma$.

This is easily checked using the facts that

$$\mathscr{L}=(\alpha/2)\Box(\Psi^*\Psi) \tag{2.6}$$

provided that the field equation (2.4) holds, and that

$$\mathrm{Re}[(D_\mu\Psi)^*(D_\nu\Psi)]=-\mathrm{Re}[\Psi^*(D_\mu D_\nu\Psi)]+(1/2)\partial_\mu\partial_\nu(\Psi^*\Psi). \tag{2.7}$$

The last equation becomes obvious if the complex function Ψ is represented in the form

$$\Psi = B\,\mathrm{e}^{\mathrm{i}\varphi} \tag{2.8}$$

where B is the amplitude and φ is the phase of Ψ, B and φ being real function.

The second step is the application of the inverse Fourier transform

$$\Psi(x) = (2\pi)^{-4}\int \Phi(k)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}^4k \tag{2.9}$$

where $\Phi(k)$ is the Fourier transform of $\Psi(x)$ and $kx = k_\mu x^\mu$, that leads to

$$-\mathrm{Re}[\Psi^*(D_\mu D_\nu\Psi)] = 2(2\pi)^{-4}\int \mathrm{Re}[\Psi^*(x)\Phi(k)\,\mathrm{e}^{\mathrm{i}kx}K_\mu K_\nu\,\mathrm{d}^4k. \tag{2.10}$$

Here $K_\mu = k_\mu - eA_\mu$, $\mu = 0, 1, 2, 3$. No mass-shell restriction on the four-vector k_μ is implied in (2.9) (see [26]).

Consequently, the Schrödinger energy–momentum tensor can be represented as follows:

$$S_{\mu\nu}(x) = 2\alpha\int F_{\mathrm{T}}(x,k)\mathscr{C}_{\mu\nu}\,\mathrm{d}^4k \tag{2.11}$$

where

$$F_{\mathrm{T}}(x,k) = (2\pi)^{-4}\,\mathrm{Re}[\Psi^*(x)\Phi(k)\,\mathrm{e}^{\mathrm{i}kx}] \tag{2.12}$$

is the eight-dimensional Terletsky distribution function and

$$\mathscr{C}_{\mu\nu} = K_\mu K_\nu + (k_\mu k_\nu - g_{\mu\nu}k_\sigma k^\sigma/2)/2 \tag{2.13}$$

is a tensor depending on the wavevectors k_μ and the external field A_μ.

In this way we obtain the following phase space representation of the Schrödinger energy–momentum tensor

$$S_{\mu\nu}(x,k) = F_{\mathrm{T}}(x,k)\mathscr{C}_{\mu\nu}(x,k). \tag{2.14}$$

In order to apply the re-interpretation principle in special relativity [15] we also need the phase space representation of the charge current of the scalar field [18]

$$\mathscr{T}_\mu(x) = i\{\Psi^*\partial_\mu\Psi - \Psi\partial_\mu\Psi^*) \qquad \mu = 0, 1, 2, 3. \tag{2.15}$$

Inserting (2.9) under the derivatives in (2.15) and omitting the integration over k we obtain immediately

$$\mathscr{T}_\mu(x,k) = -2k_\mu F_{\mathrm{T}}(x,k). \tag{2.16}$$

3. Terletsky distribution function for hydrogen-type solutions of the Klein–Gordon equation

Here we shall find the explicit expression of the function $F_T(x, k)$ defined in (2.12) for the case of hydrogen-type solutions of equation (2.4):

$$\Psi(r, \theta, \varphi, x^0) = f(r)\mathcal{P}_{l-m}(r, \theta, \varphi)\, e^{-ik_0x^0} \tag{3.1}$$

where

$$f(r) = N\beta^{3/2}\, e^{-\beta r/2}(\beta r)^m \mathcal{J}(-n+l+1, 2l+2; \beta r) \tag{3.2}$$

and

$$\mathcal{P}_{l-m}(r, \theta, \varphi) = (\beta r)^{l-m} Y_{lm}(\theta, \varphi). \tag{3.3}$$

(see [27] for details).

In the above formulae the following notation has been used: (r, θ, φ) are the spherical coordinates in $\mathbb{R}^3$; $\mathcal{P}_{l-m}(.,.,.)$ is the space spherical harmonic of degree $i-m$, $\mathcal{J}(.,.,.)$ is the degenerate hypergeometrical polynomial [28], $\beta = h/(4\pi^2 m_0 e)$, N is a normalization constant, m_0 is the rest mass of the electron, $n = 1, 2, 3, \ldots$ is the main quantum number, l is the orbital quantum number, m is the magnetic quantum number.

We shall use the following representation of the degenerate hypergeometrical polynomial [27]

$$\mathcal{J}(-n+l+1, 2l+2; \beta r) = \sum_{s=0}^{n-l-1} C_{ls}^{n}(\beta r)^s \tag{3.4}$$

where

$$C_{ls}^{n} = (-1)^s \binom{n-l-1}{s} \frac{(2l+1)!}{(2l+s+1)!}$$

The Fourier transform of the function (3.1) contains a Dirac δ-function because of its stationarity. The partial Fourier transform in $\mathbb{R}^3$ can be found applying Theorem (3.10) from Stein and Weiss [21], i.e. we have that

$$\Phi(k) = \delta(k_0 - \mathrm{k}_0) H_{3/2+j}(k_r)\mathcal{P}_{l-m}(k_r, k_\theta, k_\varphi) \tag{3.5}$$

where $(k_r, k_\theta, k_\varphi)$ are the spherical coordinates in momentum space $\mathbb{R}^3$, $j = l-m$, and

$$H_{3/2+j}(k_r) = 2\pi\, i^{m-l} \int_0^\infty f(r)(J_{1/2+j}(rk_r)/k_r^{(l+j)}) r^{3/2+j}\, dr \tag{3.6}$$

is the Hankel transform of the function $f(r)$ from (3.2). Here $J_{1/2+j}(.)$ is a Bessel function of order $1/2+j$ (see [28]).

The explicit form of the function $H_{3/2+j}(k_r)$ is obtained by the application of formula (16), Chapter 7 in [28]

$$H_{3/2+j}(k_r) = \mathcal{N}(\beta) \sum_{s=0}^{n-l-1} \frac{D_{lms}^{n}(\beta)}{(\beta^2 + 4k_r^2)^{2+2l-(m-s)/2}}\, {}_2\mathcal{F}_1\left(a, b, c; \frac{4k_r^2}{\beta^2 + 4k_r^2}\right). \tag{3.7}$$

Here the following notation is used:

$$\mathcal{N}(\beta) = 2\pi N(2\beta)^{5/2+m}(2/i)^{l-m} \qquad \text{and} \qquad D_{lms}^{n}(\beta) = C_{ls}^{n}\,(2\beta)^s \frac{\Gamma(3+2l-m+s)}{\Gamma(3/2+l-m)}.$$

$\mathcal{N}(\beta)$ and D^n_{lms} are two constants depending on the parameter β, $\Gamma(.)$ is the gamma function (see [29], Chapter V). ${}_2\mathcal{F}_1(.\,,.\,.\,;.)$ denotes the Gauss hypergeometrical function [28]

$$ {}_2\mathcal{F}_1(a,b,c;z)=\sum_{p=0}^{\infty}\frac{\Gamma(a+p)\Gamma(b+p)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+p)}\frac{z^p}{p!} \tag{3.8} $$

where $a=2+l-(m-s)/2$, $b=(-m-s)/2$, $c=l-m+3/2$ in our case.

One can easily prove that the inverse Fourier transform of (3.5) restores the wavefunction (3.1) by means of theorem (3.10) from [21] and the following integral identity (see [28])

$$ f(x)=\int_0^\infty J_\nu(tx)t\int_0^\infty J_\nu(ty)f(y)y\,\mathrm{d}y\,\mathrm{d}t. \tag{3.9} $$

Hence, from the above discussion it follows that the six-dimensional Terltsky distribution function for a hydrogen-type atom is given by

$$ F_T=Cf(r)H_{3/2+j}(k_r)\frac{J_{1/2+l-m}(rk_r)}{(rk_r)^{1/2}}Y^*_{lm}(\theta,\varphi)Y_{lm}(k_\theta,k_\varphi) \tag{3.10} $$

where C is a normalizing constant. Here $Y_{lm}(.\,,.)$ is a spherical harmonic of order $l-m$. The Bessel function $J_{1/2+l-m}(.)$ can be represented as follows (see [29], Chapter XIII)

$$ J_{1/2+l-m}(z)=(2/\pi)^{1/2}z^{l-m+1/2}(-z^{-1}\,\mathrm{d}/\mathrm{d}z)^{l-m}[(\sin z)/z]. \tag{3.11} $$

It is easy to see that this is an elementary function which converges to zero when $z\to\infty$. Also the functions $f(r)$ and $H_{3/2+j}(k_r)$ converge to zero when their arguments tend to infinity. Thus we conclude that $F_T(x,k)$ is a bounded function at infinity.

Here we want to point out that the factor

$$ \left[\frac{J_{1/2+l-m}(rk_r)}{(rk_r)^{1/2}}\right] $$

is responsible for the validity of the Parceval equality in the case of Fourier transforms in spherical coordinates both in position and momentum space.

In the particular case of the fundamental state of a hydrogen–type atom the function (3.10) was obtained by Evtimova [30]. The numerical investigation of the distribution function (3.10) in this case shows that the negative values are 0.445% of all values of this function (see [13]).

4. Discussion

Using the energy–momentum tensor of a field one adopts 'the relativistic framework in which fundamental particles are regarded as extended objects' [31], i.e. as objects with internal structure.

We next attempt to specify the possible internal constituents of a scalar particle.

As has been shown in section 2 the Schrödinger energy–momentum tensor can be transformed into a phase space density

$$ S_{\mu\nu}(x,k)=F_T(x,k)[K_\mu K_\nu+(k_\mu k_\nu-g_{\mu\nu}k_\sigma k^\sigma/2)/2] \tag{4.1} $$

where $K_\mu = k_\mu - eA_\mu$ and $F_T(x, k)$ is given by equation (2.12). It is important to notice that the Terletsky distribution function appears in (4.1) in a completely natural way through the Fourier expansion.

Now we claim that it is not difficult to interpret the negative values of the Terletsky distribution function in view of the fact that they participate in the following phase space tensor

$$\mathrm{Re}[\Psi^*(x)\Phi(k)\, e^{ikx}][3k_\mu k_\nu/2 - g_{\mu\nu}k_\sigma k^\sigma/4 + e^2A_\mu A_\nu - ek_\mu A_\nu - eA_\mu k_\nu]. \quad (4.2)$$

Here we shall exploit the idea suggested by Bohm and Vigier [32] to consider the quantum objects as specific quantum fluids. The concept that the energy–momentum tensor of a scalar field was of the same type as that of an ideal fluid was specially underlined by Anastassov [25]. Landau [33] was the first to apply the idea of ideal fluid hydrodynamics to multiparticle production by fast particle collisions. His approach was generalized by Cooper and Sharp [19] to the case of the energy–momentum tensor of interacting scalar particles, expanded into the phase space through the Wigner distribution function. The non-interacting part of this phase space tensor takes the form of a tensor of an ideal fluid.

The novelty in our approach is that the tensor (4.2) has the structure of a real fluid (see [34], chapter 2) with some peculiar properties to be discussed below.

It is necessary here to further specify the terminology. When fluids corresponding to quantum objects are defined in position space [32], [33] or in momentum space [35] they are called *quantum fluids*. If the characteristics of the considered fluids are determined by a distribution function in phase space then they will be called *sub-quantum fluids*.

It is clear that quantum fluids are obtained from sub-quantum ones by means of integration either over k or x.

The main hypothesis here is that the phase space tensor (4.2) describes the constituents of a scalar quantum object. Let us see what kind of constituents could be found in (4.2).

Since the decomposition (1.1) is valid for any quantum distribution function and since by definition $F_T^\mp(x, k) \geqslant 0$, it is clear that the phase space tensor $S_{\mu\nu}(x, k)$ can be written as a *difference* of the following two sub-quantum fluids

$$S_{\mu\nu}^\pm(x, k) = F_T^\mp(x, k)[3k_\mu k_\nu/2 - g_{\mu\nu}k_\sigma k^\sigma/4 + e^2A_\mu A_\nu - ek_\mu A_\nu - eA_\mu k_\nu] \quad (4.3)$$

where

$$2F_T^\mp(x, k) = (2\pi)^{-4}\{|\mathrm{Re}[\Psi^*(x)\Phi(k)\, e^{ikx}]| \pm \mathrm{Re}[\Psi^*(x)\Phi(k)\, e^{ikx}]\} \quad (4.4)$$

Combining the idea suggested by Vigier and Terletsky [16] with the considerations of Pavšič and Recami [31] we shall juxtapose 'constitients' to $S_{\mu\nu}^+(x, k)$ and 'anti-constituents' to $S_{\mu\nu}^-(x, k)$. In other words, the representation (4.3) allows assumption of the existence of at least two streams of 'constituents' and 'anti-constituents' in a hypothetical sub-quantum level of the matter described in phase space. Here by sub-quantum we mean the level that precedes the quantum one, i.e. it is connected with the internal structure of any elementary quantum object.

Since one has that

$$S_{\mu\nu}(x, k) = S_{\mu\nu}^+(x, k) - S_{\mu\nu}^-(x, k) \quad (4.5)$$

one can say that each negative 'anti-constituent' $-S_{\mu\nu}^-(x, k) = -F_T^-(x, k)\, \mathscr{C}_{\mu\nu}(x, k)$ annihilates (figuratively said 'eats up') the action of a corresponding 'constituent'

$S^{+}_{\mu\nu}(x,k)=F^{\mp}_{\mathrm{T}}(x,k)\mathscr{C}_{\mu\nu}(x,k)$, so that the remaining uncompensated 'constituents' determine the four-momentum of the quantum object

$$P_{\mu}=\int S_{\mu 0}(x,k)\,\mathrm{d}^3x\,\mathrm{d}^4k. \tag{4.6}$$

Next we shall consider the characteristics of the above introduced sub-quantum fluids. For example, the local pressures and the local energy densities of these fluids are equal to

$$p^{\pm}(x,k)=(1/4)F^{\mp}_{\mathrm{T}}(x,k)k_{\sigma}k^{\sigma} \tag{4.7}$$

$$\rho^{\pm}(x,k)=(5/4)F^{\mp}_{\mathrm{T}}(x,k)k_{\sigma}k^{\sigma} \tag{4.8}$$

provided that $k_{\sigma}k^{\sigma}\neq 0$. Hence, the local four-velocities of the two sub-quantum fluids are given by

$$u_{\mu}=k_{\mu}/(|k_{\sigma}k^{\sigma}|)^{1/2} \qquad k_{\sigma}k^{\sigma}\neq 0 \qquad \mu=0,1,3,4. \tag{4.9}$$

The terms responsible for the interactions are as follows

$$\Delta S^{\pm}_{\mu\nu}(x,k)=F^{\mp}_{\mathrm{T}}(x,k)(e^2A_{\mu}A_{\nu}-ek_{\mu}A_{\nu}-eA_{\mu}k_{\nu}). \tag{4.10}$$

From this representation it is clear that the electromagnetic potential A_{μ} acts directly on the wave four-vectors k_{ν}, $(\mu,\nu=0.1,2.3)$ in the Fourier expansion of the wavefunction. Also a self-interaction of the electromagnetic field is included.

Now we shall direct our attention to some of the peculiarities of the above energy–momentum tensors in phase space.

First, it is obvious from equations (4.7) and (4.8) that their local pressures and local energy densities can be positive as well as negative since both cases $k_{\sigma}k^{\sigma}>0$ and $k_{\sigma}k^{\sigma}<0$ are possible. Also, sub-quantum fluids without local pressure and local energy are included: $k_{\sigma}k^{\sigma}=0$.

Second, since these hypothetical sub-quantum fluids contain time-like $(u_{\sigma}u^{\sigma}=-1)$, space-like $(u_{\sigma}u^{\sigma}=-1)$ and isotropic $(u_{\sigma}u^{\sigma}=0)$ four-velocities, they consist of bradyons, tachyons and luxons (see Recami [38]).

The question about the existence of tachyons (faster-than-light objects) has been discussed by many authors from different points of view [37–44].

Now we shall apply the reinterpretation principle in special relativity (see [15] and [45]) to our case. For this purpose we shall consider the zeroth component of the current in phase space

$$\mathscr{J}_0(x,k)=-k_0[F^{+}_{\mathrm{T}}(x,k)-F^{-}_{\mathrm{T}}(x,k)]. \tag{4.11}$$

Since $\mathscr{J}_0(x,k)$ depends on the charge of the scalar field it is obvious that the change of the energy sign $(k_0=E/hc)$: $k_0=\leftrightarrow(-k_0)$ means a change of the charge sign of the 'constituents'/'anti-constituents' of the sub-quantum fluids. Also, as Recami [45] pointed out, a change of the energy sign in momentum space when using Fourier expansions in the field theory corresponds to a change in the sign of time in the coordinate (dual) space and vice versa, i.e. $k_0\leftrightarrow(-k_0)$ means $x^0\leftrightarrow(-x^0)$. Thus, 'one can reinterpret any negative energy object P (travelling backward in time) in terms of its anti-object $\bar{P}$ going the opposite way (endowed with positive energy and travelling forward in time)' [45]). This leads to the conclusion that 'constituents'/'anti-constituents' of sub-quantum fluids consist of bradyons, tachyons and luxons as well as anti-bradyons, anti-tachyons and anti-luxons.

5. Application

Let us consider briefly the above-introduced local characteristics of the real sub-quantum fluid related to the fundamental state of the hydrogen-type atom. The Terletsky distribution function in this case depends only on the radial variables in $\mathbb{R}^3 \times \mathbb{R}^3$ (see [26])

$$F_{\mathrm{T}}(x, k) = \delta(k_0 - k_0)\, \mathrm{e}^{-\imath k_0 x^0} \frac{N\, \mathrm{e}^{-\beta r/2} \sin(rk_r)}{(4k_r^2 + \beta^2)^{3/2}(rk_r)}\, Q(k_r) \tag{5.1}$$

where the function $Q(k_r)$ is given by the expansion

$$Q(k_r) = \sum_{p=0}^{\infty} \frac{\Gamma(p - 1/2)}{p!} \left(\frac{4k_r^2}{4k_r^2 + \beta^2}\right)^p \tag{5.2}$$

and N is a normalizing constant.

The metric in Minkowski space is as follows

$$g_{\mu\nu} = \mathrm{diag}(1, -1, -r^2, -r^2 \sin^2\theta). \tag{5.3}$$

Hence the square of the wavevector will be

$$k_\sigma k^\sigma = k_0^2 - k_r^2 - r^2 k_r^2 - r^2 \sin^2\theta\, k_\varphi^2. \tag{5.4}$$

Thus, it is clear that the local pressures (4.7) and the local energy densities (4.8) depend on all the components of the wavevectors k_μ, participating in the Fourier expansion (2.9). Therefore, from the relations (4.7), (4.8), (5.1) and (5.4) one concludes that the sub-quantum fluid (4.1) of the hydrogen-type atom shows dispersion in the pressure and energy density even in the fundamental state.

Since the external electromagnetic potential is equal to the potential of the proton at the centre of the hydrogen atom: $A_\mu = (e/r,\ 0,\ 0,\ 0)$ we see that the tensor responsible for the interaction has the following components

$$\Delta S_{00}^{\pm} = F_{\mathrm{T}}^{\pm}(x^0, r, k_0, k_r)[(e^2/r^2) - (2ek_0/r)] \tag{5.5a}$$

$$\Delta S_{0r}^{\pm} = -F_{\mathrm{T}}^{\pm}(x^0, r, k_0, k_r)[(ek_r/r)] \tag{5.5b}$$

$$\Delta S_{0\theta}^{\pm} = -F_{\mathrm{T}}^{\pm}(x^0, r, k_0, k_r)[(ek_\theta/r)] \tag{5.5c}$$

$$\Delta S_{0\varphi}^{\pm} = -F_{\mathrm{T}}^{\pm}(x^0, r, k_0, k_r)[(ek_\varphi/r)] \tag{5.5d}$$

which differ from zero. The remaining components of $\Delta S_{\mu\nu}$ for $\mu, \nu = r, \theta, \varphi$ are equal to zero.

Here it is obvious that the longitudinal interaction plays role in $\Delta S_{0r}^{\pm}$ and the transverse interaction is important in $\Delta S_{0\theta}^{\pm}$ and $\Delta S_{0\varphi}^{\pm}$.

6. Concluding remarks

In this paper we have included the one-particle Terletsky distribution function of a scalar quantum object in the hydrogen-type atom in the phase space representation of the Schrödinger energy–momentum tensor. This fact reveals an opportunity for the interpretation of the negative values of the distribution functions. Any quantum object could be considered as composed of some kind of sub-quantum entities defined in phase space. These sub-quantum objects have the structure of fluids with positive

($F_T^+(x,k) \geqslant 0$) as well as negative ($-F_T^-(x,k) \leqslant 0$) densities in phase space. The species with positive density ($S_{\mu\nu}^+(x,k)$) are considered as 'constituents' and those with negative density ($-S_{\mu\nu}^-(x,k)$) as 'anti-constituents' of the sub-quantum fluid $S_{\mu\nu}(x,k)$ (see 4.3) and (4.5)). The latter one precedes the quantum fluid $S_{\mu\nu}(x)$ (2.1) (for the terminology see [31]). Since no mass-shell restriction on the four-momentum k_μ is imposed new types of bradyon, tachyon and luxon streams in the 'constituent' and 'anti-constituent' fluids $S_{\mu\nu}^+(x,k)$ and $S_{\mu\nu}^-(x,k)$ appear. Also, as the re-interpretation principle in special relativity [15], [45] has been proved valid in our case, the phase space representation of the energy–momentum tensor (4.2) contains anti-bradyon, anti-tachyon and anti-luxon fluids as well. The tachyonic/anti-tachyonic interpretation of the distribution functions is in agreement with Kruger's suggestion [14]. The quantity of the negative values ($-F_T^-(x,k) \leqslant 0$) in the total distribution function $F_T(x,k)$ is very small: it amounts barely to 0.445% of all the values of the distribution function in the fundamental state of the hydrogen-type atom.

It is necessary to underline that simultaneously definite position and momentum of the quantum particle are not supposed to take place.

Here we want to point out that all the above statements are open for discussion. The given interpretation is biased: it describes the quantum objects as composed of sub-quantum fluids, i.e. the above picture manifests only the continuous character of the quantum objects, but their discontinuity is not taken into account.

Finally we want to mention that there always exists a possibility to model any fluid by means of infinitely many chaotically moving fluxes of 'particles' and 'anti-particles' with suitable properties. What kind of properties are needed for these hypothetical sub-quantum objects will be discussed in a forthcoming paper.

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